Bull. Spec. Astrophys. Obs., 2003, 56, 46-56

A convective model of differential rotation of the Sun

L. G. Kaplan[a,b]

[a] Stavropol State University, 360009, Stavropol, Pushkina st., 1
[b] Special Astrophysical Observatory of the Russian AS, Nizhnij Arkhyz 369167, Russia
l_kaplan@mail.ru

Received July 18, 2003; accepted July 28, 2003.

Abstract. A model of convection in the upper layers of the Sun through gaseous bubbles of spherical shape is considered. Simplified assumptions are made of thermodynamic spherical isotropy of the Sun, a sharp boundary between the radiative part and the convective layer, absence of friction and involving of surrounding gas into the bubbles under their movement. It is shown that when emerging in the layer of turbulent gas, the bubble rotates about the axis running through its center parallel to the polar axis of the Sun. When emerging, the bubble is flown circularly past by the surrounding gas and the Zhukovsky (Magnus) force come into play, and, because of the rotation of the Sun, the Coriolis force also begins to work. The combined effect of these forces cause acceleration of the bubble in the direction of rotation of the Sun. As the bubbles dissipate in the outer regions of the Sun, these regions begin to rotate faster with respect to its central part, the acceleration gained being greater at the equator in comparison with that at the poles. The differential rotation effect is shown to rise with increasing density of the heat flow, of the distance passed by the bubbles from the moment they originated to dissipation, of the thickness of the convective layer and to decrease with increasing dynamic friction coefficient. The derived theoretical results agree with the observed pattern of differential rotation of the visible surface of the Sun.

Key words: Sun: rotation – convection

1. Introduction

The differential rotation of the Sun has been established reliably (Tossul 1982). The equatorial region rotates faster than the high latitude solar surfaces. The empirical relationships derived by different authors are distinguished insignificantly. From the known formula of Howard and Harvy (Tossul, 1982) the round-the-clock revolution of the surface

$$\varepsilon(\psi) = 13,76 - 1,74Sin^2\psi - 2,19Sin^4\psi\,[^\circ/24^h] \quad (1)$$

in the equatorial region is 1.4 times as large as in the polar regions (ψ is the heliocentric latitude). The differential rotation is characteristic of many other stars and also of Jupiter.

Originally a two-dimensional model of different rotation was proposed. According to this model the volume was divided by conic surfaces corresponding to the unchanged heliocentric latitude (Fig. 1). Further it was supposed that the natural convection (under the action of the buoyancy force) is realized through the bubbles heated more than the surrounding gas. The bubble, as it floats up, remains inside the original conic surface since the buoyancy force is directed along the radius. That is why, convective flows of the neighboring cones do not interact and the meridional gas flows are absent. The babble preserves the angular momentum and therefore its tangential velocity (circular in the latitudinal direction in the inertial frame) decreases as it floats up (Fig.2). As the bubble dissipates at the surface of the Sun, the angular momentum is transferred from the bubble to the surrounding gas, which is accompanied by deceleration of rotation of the outer layers. This decrease is greater in the equatorial region than in the polar regions. Allowance for the Coriolis force yields the same result when treating the movement of the bubble in the rotating coordinate system associated with the solid-body rotating central part of the Sun.

However, the opposite is observed: the angular rotational velocity of the equatorial region of the Sun is higher than that of the polar regions. Because of this the two-dimensional model was recognized as an inadequate one and a number of models of differential rotation were suggested. To explain the real pattern in these models, there were considered, for instance, the supposed meridional gas flows deep in the Sun and at its surface, strong magnetic fields (Vandakurov 1999), particular character of turbulence etc. The hy-

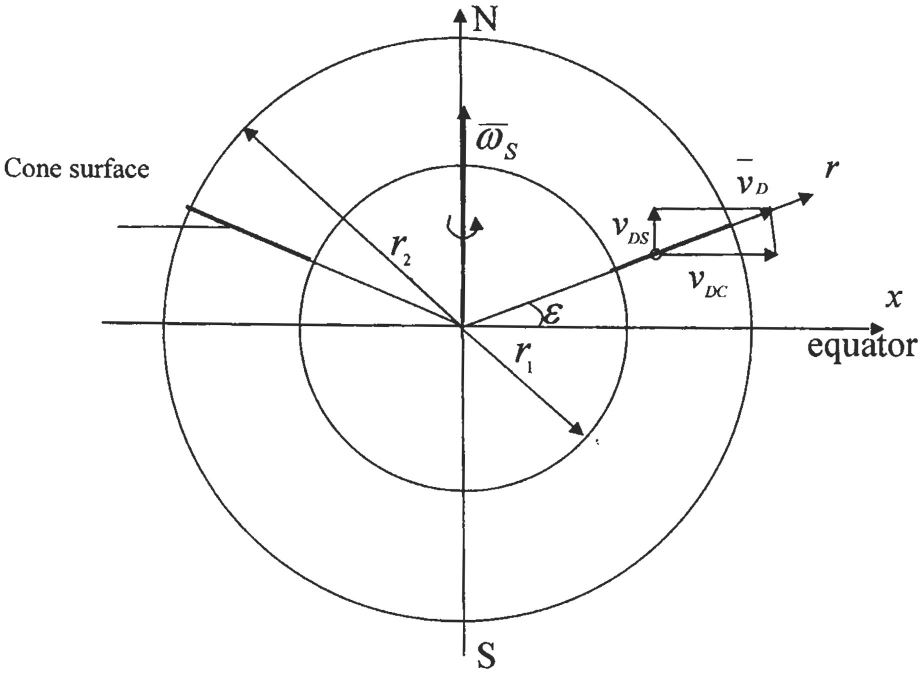

Figure 1: *The meridional section of the Sun.*

drodynamic substantiation of these models is not convincing enough. Observational data suggest that the meridional flows are not significant at the surface of the Sun.

A refined two-dimensional model of thermodynamics of the upper layers of the Sun is proposed in this paper, which explains the accelerated rotation of its surface in the equatorial region in comparison with the polar ones.

2. Original propositions and assumptions

Some known facts, adopted notions and simplified assumptions concerning the structure and parameters of the Sun having a compromise character are presented below. The shape of the Sun is adopted to be spherical, its ellipticity due to rotation is not taken into account. The central, non-convective part of the Sun (its radius is r_1, in which nearly all its mass is concentrated (Fig. 1) rotates as a solid body, the vector of the angular velocity $\overline{\omega_s}$ is directed northward. The dynamic processes in the upper layers of the Sun have practically no effects on its central part. The statistical characteristics of the thermodynamic processes on the Sun are not time-variable. The Sun is spherically uniform and its heat capacity c_p, temperature T_e, specific heat flow q, as well as other thermodynamic characteristics do not depend on the heliocentric coordinates.

Energy is generated in the central part of the Sun and transmitted to the lower boundary of the convective layer by radiative transfer. The upper convection layer r_2 is equal to the visible radius of the Sun. The depth of the convective layer is the same everywhere, independent of the heliocentric latitude and is not large, $\Delta r \equiv r_2 - r_1 << r_2$.

In convection the heated gaseous bubbles float up to the surface of the Sun, and the surrounding gas settles down on their place. The gas temperature inside the bubble is higher ($T_i > T_e$), while the density is lower ($\rho_i < \rho_e$) than the corresponding parameters of the surrounding gas at the same height. A bubble originating at a distance R from the center of the Sun ($R > r_1$) passes then a certain way l in the radial direction and dissipates at a distance $R + l$ from the center. We believe that the l is the same for all bubbles and not large ($l << \Delta r$). The initial size and mass of the bubbles are such that they are practically isolated from the surrounding gas, and in the process of emergence no considerable change in their masses and temperatures occurs as a result of involving of surrounding gas into the bubbles and heat exchange of bubbles with the surrounding gas. The expansion of the bubbles as they emerge and the adiabatic cooling accompanying this expansion do not affect the effects we are interested in. The dynamic equilibrium of the bubble moving at a constant speed in the surrounding gas is established by its shape close to a sphere (Kaplan, Nesis 1998). Thus, to a quasi-stationary ap-

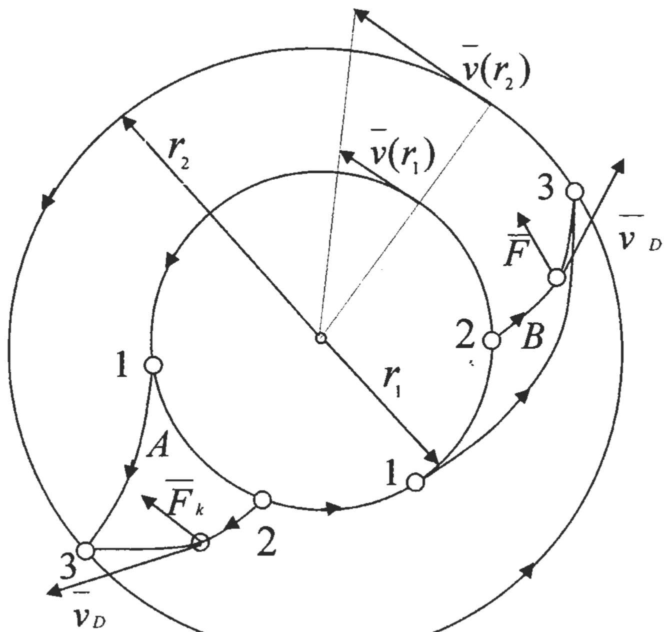

Figure 2: *The equatorial section of the Sun; the trajectory of the bubbles from the previous (A) and suggested (B) models in the inertial (B) and rotating coordinate system.*

proximation, the bubbles are spherically symmetric. Assume that the radius of the bubbles is smaller than the depth of the convective layer, $R_s < \Delta r$, so the gas density at its surface is practically unchangeable.

In consequence of friction the vorticity of the surrounding gas (Loitsyansky 1987) is transmitted to the bubble as it forms and moves. Owing to this, the bubble rotates about the polar axis, running through the center, at an angular velocity equal to the angular velocity of the surrounding gas. In turn, the rotation favors equalization of temperature inside the bubble and its thermal homogeneity (internal barotropy). In conjunction with the sphericity the thermal homogeneity leads to non-orientation of the bubble.

Under the action of the buoyancy force the bubble acquires high speed, but this speed is lower than that of sound. The movement of the bubble at this velocity does not cause significant changes in the gas density in the region of flowing around and its surface either, as compared to the free medium at the same height (Kaplan 1993; Loitsyansky 1987). The forces affecting the bubble in a tangential direction are weaker than those affecting in a vertical direction. This is why, despite the action of these forces, the direction of the bubble movement is close to radial when observing in the rotating coordinate system connected with the layer of the gas surrounding the babble. The bubble, when floating up, remains inside the initial conic surface characterized by the unchangeable heliocentric latitude. Because of this, the general three-dimensional pattern of convection on the Sun is, in fact, an aggregate of two-dimensional patterns inside individual conic surfaces.

3. Convective movement of bubbles in the equatorial plane of the Sun

3.1. Dynamic characteristics of the gas

First, we consider a one-dimensional model of rotating gas in the equatorial section of the Sun. Here we use the known relationships (Loitsyansky 1987). The gas in the plane of the section moves in the inertial frame at a velocity $\nu(r)$, which is dependent on the distance to the center r and independent on the angle. Since the radial velocities have only polar components in this case, they may be represented as scalar quantities. The angular global velocity of rotation of the gas layer located at the distance r near the center of the Sun is equal to

$$\omega_g(r) = \frac{v(r)}{r}. \tag{2}$$

The angular local velocity of rotation of matter in this layer is

$$\omega_l(r) = \frac{1}{2} rot\ v(r) = \frac{1}{2r}\left(\frac{d(v(r)r)}{dr}\right) = \frac{1}{2}\left(\frac{v(r)}{r} + \frac{dv(r)}{dr}\right). \tag{3}$$

It follows from (2, 3) that

$$\frac{dv(r)}{dr} = 2\omega_l(r) - \omega_g(r);$$
$$\omega_l(r) = \omega_g(r) + \frac{1}{2}r\frac{d\omega_g(r)}{dr}. \quad (4)$$

In the coordinate system related to the mentioned layer and rotating at an angular velocity $\omega_g(r)$ the gas rotates at the angular velocity $\omega_l(r) - \omega_g(r)$. At $\frac{v}{r} = \frac{dv}{dr}$, $\omega_g(r) = \omega_l(r)$ the gas rotation in the layer is that of a solid body, while at $\frac{v}{r} = -\frac{dv}{dr}$, $\omega_l(r) = 0$ it is vortex-free. The gas rotates as a solid body in the central part of the Sun ($r < r_1$), where $\omega_g(r) = \omega_s = Const$. The mutual sliding of the gas layers is characterized by the component of the tensor of the deformation rates

$$def\ v(r) = \frac{1}{2}(\frac{dv(r)}{dr} - \frac{v(r)}{r}) =$$
$$= \frac{1}{2}r\frac{d\omega_g(r)}{dr} = \omega_l(r) - \omega_g(r). \quad (5)$$

Deformation is absent at solid-body rotation, when $\omega_l(r) = \omega_g(r)$.

3.2. The theory of a way of mixture

The theory of a way of mixture of Prandtl (Loitsyansky 1987) is generally used in calculating turbulent stresses arising at stratified flow of liquid and forced convection. We will make use of this method for calculating stresses springing up with natural convection under the action of gravity. Replace the continuous distribution of the bubbles by stepwise (Fig. 2, 3). Compare the way of mixture of Prandtl with the way from origin to dissipation of the bubble. Consider a certain gas layer located between the circumferences R, $R + l$ in the region of convection ($r_1 < R < r_2$). The angular velocities of rotation of matter $\omega_l(r)$ and the layer as a whole $\omega_g(r)$ change in a thin layer ($l < \Delta r$) insignificantly and are assumed to be equal to the quantities at the lower boundary of the layer ($\omega_g \equiv \omega_g(R) \cong \omega_g(r)$, $\omega_l \equiv \omega_l(R) \cong \omega_l(r)$).

To the lower boundary ($r = R$) refer the bubbles originating at a distance of $\pm l/2$ from it. To the upper boundary ($r = R + l$) refer the bubbles dissipating at a distance $\pm l/2$. Suppose that the processes of formation, emergence and dissipation of the bubbles are time separated. The bubbles first form completely at the lower boundary of the layer and only after that they begin to float up. Their dissipation occurs at the upper boundary. Interpret the movement of the bubble as a strongly localized process (Kaplan 1993; 2000). In this local process the bubble itself is a nucleus, and the region around it, where the downward (replacing) motion of the gas occurs as the bubble moves upward, is the pass-over region. It has no fixed boundaries with the external gaseous medium. Friction and power interaction of the region encompassed by the local process with the gas layer outside of the pass-over region are absent.

Introduce a reference system with the origin at the center of the Sun, which rotates with the angular frequency of the layer ω_g together with the gas located at the lower boundary of the layer being discussed ($r = R$). The initial tangential velocity of the bubble in this coordinate system, as well as that of the surrounding gas, is equal to zero ($v(R) = 0$). The velocity of the gas at the upper boundary of the layer ($r = R + l$) is

$$v_r(R + l) = (\frac{dv}{dr} - \omega_g)\,l = 2(\omega_l - \omega_g)l, \quad (6)$$

where (4) is taken into account.

3.3. Mass and heat transfer by means of the bubbles

Consider one of the bubbles which has a volume B and a mass

$$m_i = \int_B \rho_i\, dV. \quad (7)$$

In accordance with the assumption made, the volume B and the mass m, the gas density ρ_i and temperature T_i inside the bubble and also the thermodynamic characteristics of the surrounding gas are believed to be independent of the distance to the center of the Sun (or may be reduced to the lower boundary of the layer).

In the same volume of space outside the bubble the mass is equal to

$$m_e = \int_B \rho_e\, dV. \quad (8)$$

The differential mass is

$$m_\Delta = m_i - m_e = \int_B \rho_\Delta\, dV, \quad (9)$$

where $\rho_\Delta = \rho_i - \rho_e \approx -\rho_i\frac{\Delta T}{T_e}$ is the difference of gas densities inside and outside the bubble, $\Delta T = T_i - T_e$. It is obvious that the differential mass of the bubble is negative.

The process of mass transfer in the layer can be represented by the example of one bubble. The heating of the gaseous bubble as it forms causes outflow of excess mass $|m_\Delta|$ from its volume. This excess mass is transferred from the lower to the upper boundary of the layer since the masses of the central part of the Sun and of the layer discussed remain unchanged (Fig. 3). Then the emergence of a bubble of mass m_i from the lower to the upper boundary of the layer occurs. The gas in the pass-over region occupies the

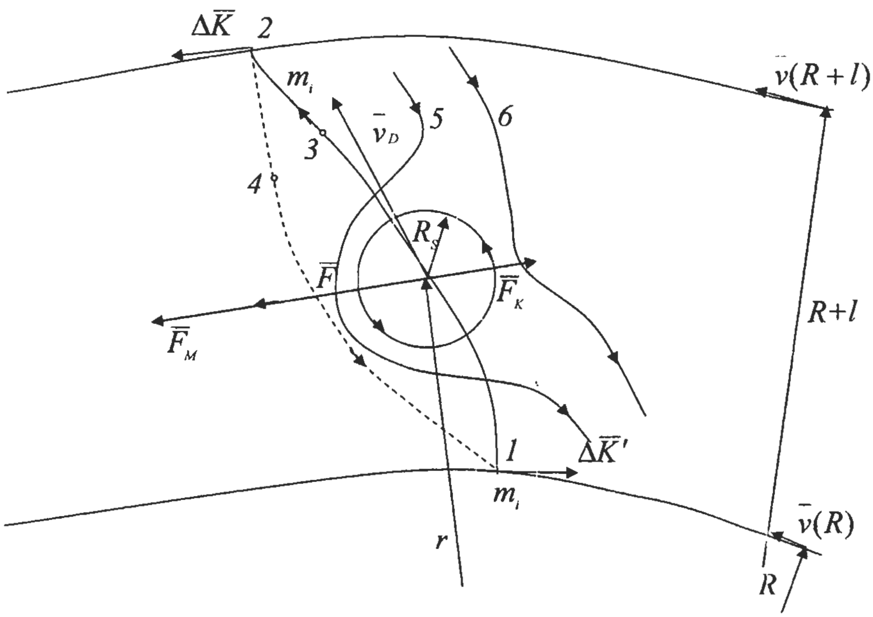

Figure 3: *The layer of mixing: the levels of origin – (1) and dissipation – (2); 1-3-2 — the trajectory of the bubble; 2-4-1 — the conditional trajectory of the gas in the pass-over region; 5, 4, 6 — the lines of the flow.*

volume taken up earlier by the bubble, and the mass m_e sinks from the upper to the lower boundary. Then equalization of temperatures between the bubble and the surrounding gas takes place. In this process the excess mass $|m_\Delta|$ carried away from the bubble when it formed comes to the bubble volume. After this, dissipation of the bubble is completed and it ceases to exist, losing individual characteristics.

Taking into account that a great number of bubbles take part in convection, one can somewhat simplify the represented pattern: unite the processes of formation, emergence and dissipation and consider that when the bubble of mass m_i floats up, the same mass of gas sinks in the region of passing over, $m_i = m_e - |m_\Delta|$. Thus, as the bubble crosses the layer being treated from bottom upwards, these boundaries are crossed simultaneously by the same mass of gas in the pass-over region from top to bottom.

In the process of convection the following amount of heat is driven to the upper boundary of the layer:

$$Q_l = c_p m_i \Delta \mathrm{T} = -c_p m_\Delta \mathrm{T}_e, \qquad (10)$$

where $\Delta T = T_i - T_e$ is the difference of temperatures between the bubble and the surrounding medium, c_p is the heat capacity of the gas at constant pressure, (9) is allowed for. A unit volume is thought to contain n bubbles of average differential mass m_Δ, which float up at an average radial velocity v_D. The averaged heat flow carried out by the bubbles through a unit area during a unit time is equal to

$$q = \langle Q_l n\, v_D \rangle = -c_p T_e \langle m_\Delta n\, v_D \rangle . \qquad (11)$$

The mean specific flow of differential mass driven from here to the surface during a unit time is equal to

$$\langle m_\Delta n\, v_D \rangle = -\frac{q}{c_p T_e}. \qquad (12)$$

Designate the absolute value of this flow as $A \equiv |\langle m_\Delta n\, v_D \rangle|$.

3.4. Dynamic characteristics of the bubble

Let us consider the movement of the bubble in the rotating frame of axes connected with the lower boundary of the convection layer. The bubble in the initial position at the lower boundary of the convective layer is immovable with respect to the surrounding gas in this coordinate system. As the bubble moves in the radial direction it is affected by the buoyancy force depending on the differential mass and the force of gravity on the Sun

$$F_B = -g m_\Delta . \qquad (13)$$

When floating up under the action of the force F_B, the potential energy of gravity $-g m_\Delta l$ changes completely to the kinetic energy $\frac{m_i v_D^2}{2}$. Thus, on the way l the bubble in the process of floating up gathers a radial velocity

$$v_{D\ Max} = \sqrt{-\frac{2 m_\Delta g l}{m_i}} \quad . \qquad (14)$$

This quantity is the upper limit of the radial component of the bubble velocity. Since in radial movement the bubble rotates together with the gas

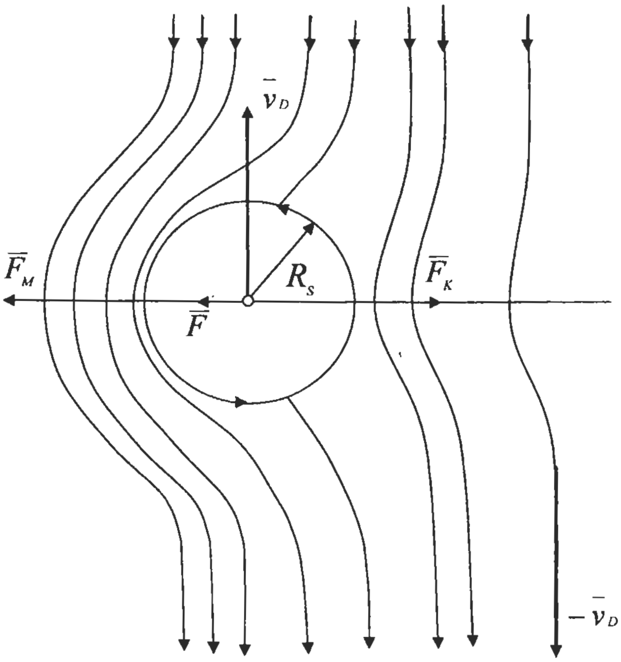

Figure 4: *Circular flow of the surrounding gas around the bubble.*

layer, then it is under the action of the Coriolis force (Fig. 3). Besides, interaction between the pass-over region gas and the bubble occurs. The tangential component of the momentum acquired by the bubble is carried to the upper boundary of the layer and transferred to the surrounding gas as the bubble dissipates.

Represent the bubble in the flat model in the form of a circle of radius $R_S < l$. Being in the gas layer at the distance r from the Sun center $(R < r < R+l)$, the bubble rotates about its center at the angular velocity ω_l together with the surrounding gas. At the same time it moves under the action of the buoyancy force at the velocity v_D in a direction close to radial.

To calculate the force with which the pass-over region affects the bubble, fix at some moment of time the inertial frame connected with the center of the circle (Fig. 4). In this frame interprete the movement in the radial direction as pass over by a homogeneous flow of the surrounding gas. Assume that the tangential component of the linear velocity of the bubble is lower than the radial velocity ($v_t << v_D$) and, therefore, the sense of the velocity vector is close to the radial direction and its value is equal to v_D. The instantaneous gas velocity field near the bubble is a superposition of two fields: the velocity field of the flow passing over it in the radial direction and the velocity field of the flow rotating around it. In circular passing over (Loitsyansky 1987), a force arises that acts in the direction of rotation at a right angle to the radial velocity vector. Origin of this force is illustrated in Fig. 4. On the left side the pass-over and circulation velocities have the same direction and are added together at $\omega_l > 0$, while on the right side they have opposite direction and are subtracted. This is why the velocity of the gas on the left side is higher than on the right. In accordance with Bernoulli's theorem, the pressure on the right side of the circle is higher than on the left, and the bubble is affected by a force in the direction of rotation. According to the known theorem of Zhukovsky the arising force is equal to

$$F_M = -\rho_e I v_S = \rho_e I v_D, \tag{15}$$

where ρ_e, I are the density of the gas and the velocity circulation at the outer boundary of the circle, respectively, v_S is the velocity of the gas relative to the circle.

Find the circulation $I = \int_L \bar{v}\, d\bar{l}$ of the gas velocity $\bar{v}$ near the contour of the area L, $d\bar{l}$ is the element of the contour. The linear velocity of the gas at the boundary of the circle is equal to $v = \omega_l R_S$. The length of the boundary is $2\pi R_S$, and the velocity circulation is

$$I = \omega_l R_S\, 2\pi R_S = 2\omega_l S. \tag{16}$$

Substituting (16) into (15) obtain

$$F_M = 2\omega_l v_D \rho_e S. \tag{17}$$

In a three-dimensional representation the force affecting the bubble depends on the thickness of the gas layer Δz (Fig. 5):

$$\begin{aligned} \Delta F_M &= 2\omega_l v_D \rho_e S \Delta z = \\ &= 2\omega_l v_D \rho_e \Delta V = 2\omega_l v_D \Delta m_e, \end{aligned} \tag{18}$$

where ΔV, Δm_e are the volume of the bubble and the mass of the replacing surrounding gas in its volume, respectively.

Treating a number of parallel sections (Fig. 5) and considering that in the process of movement the bubble rotates as a solid body, obtain from (16) the Zhukovsky force (in some papers this force is named the Magnus force) having an effect on the bubble as it is circularly passed over.

$$F_M = 2m_e\omega_e v_D, \tag{19}$$

where $m_e = \rho_e V$ is the mass of the surrounding gas in a volume equal to the bubble volume.

According to Newton's second law, the bubble affects the gas in the pass-over region with the same force, but in opposite direction.

$$F'_M = -2m_e\omega_e v_D. \tag{20}$$

The bubble rotates as a solid relative to the center of the Sun together with the gas layer in which it is situated at an angular velocity close to ω_g. Its radial movement results in appearing of the Coriolis force. This force is directed at a right angle to the direction

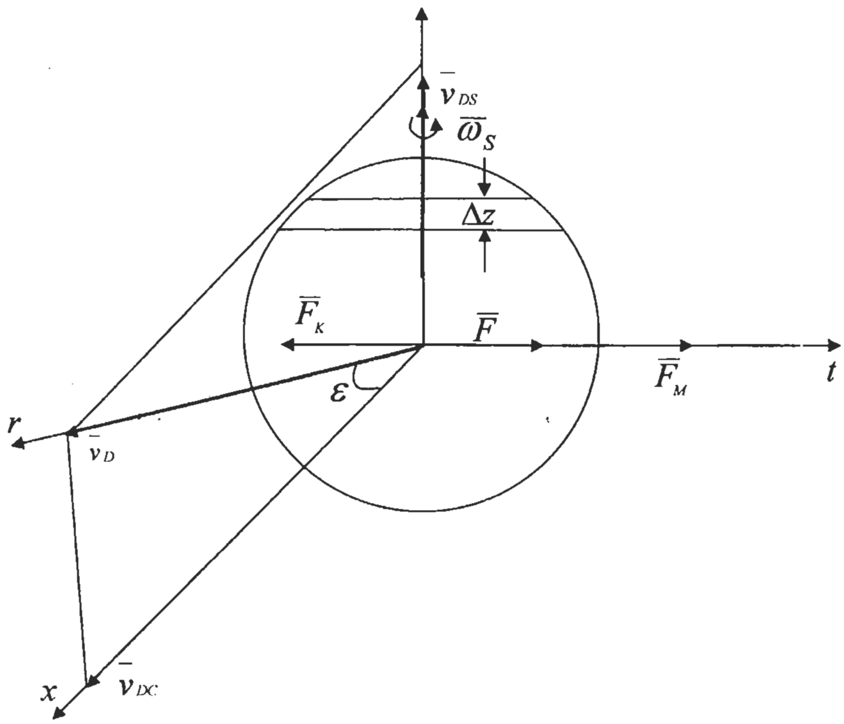

Figure 5: *Forces affecting the bubble as it moves in the rotating coordinate system and is passing over by the surrounding gas.*

of the bubble movement and in the direction opposite to the rotation of the layer, and is equal to

$$F_K = -2m_i\omega_g v_D. \tag{21}$$

In contrast to (19) the actual mass of the bubble is involved in this formula. The negative sign corresponds to the action of the force opposite to the layer rotation. When floating up of a bubble of mass m_i at a speed v_D the same mass of gas moves in the pass-over region in the opposite direction at a rate -v_D. The Coriolis force will have an effect on the sinking gas. Since the general direction of the velocity of motion of matter in the pass-over region is opposite to that of the bubble movement, and the masses are equal, then the corresponding forces are different only in sign

$$F_K^{'} = 2m_i\omega_g v_D. \tag{22}$$

Adding together (19) and (21), we obtain the component of the force affecting the bubble tangentially:

$$F = F_M + F_K = 2(\omega_l m_e - \omega_g m_i) v_D. \tag{23}$$

Adding together (20) and (22), obtain the forces having a tangential effect on the gas in the pass-over region

$$F^{'} = F_M^{'} + F_K^{'} = -2(\omega_l m_e - \omega_g m_i) v_D, \tag{24}$$

which is equal to F in value and opposite in direction. Consider the case where the rotation is that of a solid body ($\omega_l = \omega_g$) and the densities of the bubble and the surrounding gas are equal, so, $m_e = m_i$. Then, in correspondence with (23, 24) the acting forces are equal to zero. On the other hand, as the bubble moves (for instance, under its own momentum), at some place the mass of the bubble is replaced by the same mass of the surrounding gas, while in another place the reverse takes place. Neither do change other local parameters (linear and angular velocities). The potential energy of the gravitating gas of the Sun as a whole remains unchanged either, since the spatial distribution of masses has not changed. Thus, the integrated forces (Loitsyansky 1987) having an effect on the bubble and the pass-over region are equal to zero in full agreement with the obtained result.

During the time of rising T the bubble acquires the tangential component of the momentum equal to

$$K_D = \int_0^T F dt \cong 2(\omega_l m_e - \omega_g m_i)\, l. \tag{25}$$

Similarly, during the time of sinking T the gas in the pass-over region acquires the tangential component of the momentum equal to

$$K_D^{'} = \int_0^T F^{'} dt \cong -2(\omega_l m_e - \omega_g m_i)\, l. \tag{26}$$

The mass m_i, leaving the upper layers of the Sun, has a momentum

$$K_i = m_i v_r(R+l) = 2m_i(\omega_l - \omega_g)\, l. \qquad (27)$$

Thus the process of rising of the bubble and the compensating sinking of the gas having the same mass results in transport of the tangential component of the momentum to the upper layer of the star

$$\Delta K = K_D - K_i = 2(m_e - m_i)\omega_l\, l = -2m_\Delta \omega_l\, l. \quad (28)$$

As the bubble dissipates, the quantity ΔK is transferred to the gas located above the gas layer being discussed. In conformity with the above said, the gas in the pass-over region when crossing the upper boundary of the layer, gains the initial momentum determined by formula (27). In the process of sinking, (26) is added to this momentum. Finally, when crossing the lower boundary of the layer the momentum

$$\Delta K' = K_D' + K_i = 2m_\Delta \omega_l\, l. \qquad (29)$$

is carried away.

Thus, as the gas sinks in the pass-over region a momentum equal in value and opposite in sign to the momentum carried over by the emerging bubble to the upper level is driven to the lower level.

The momentum transported by the bubble to the upper level is positive and favors accelerated rotation of the upper part of the convection layer with respect to the lower part. The tangential component of the bubble velocity (relative to the upper boundary of the layer considered) is equal to

$$v_t = \Delta K/m_i = -2m_\Delta \omega_l\, l/m_i. \qquad (30)$$

From comparison of (14) and (30) it follows that the condition

$$l < \frac{m_i g}{2m_\Delta \omega_l^2}$$

is complied with.

Since $m_\Delta << m_i$, then this inequality is met on any reasonable assumptions concerning the path l. Consequently, the condition $v_t << v_D$ is always met as well.

Fig. 2 shows the movement of the bubbles in the equatorial section of the Sun in accordance with the existing model (A) and the proposed one (B). For clarity, the trajectories of the bubbles are increased up to the thickness of the convection layer, $l = \Delta r$. Point 1 is the start of the radial movement of the bubbles when observing in the inertial frame. The arcs 12 correspond to the rotation of the central part of the Sun during the time the bubble moves form the depth outwards. In the inertial frame the paths of the bubbles — 13, while in the rotating coordinate system — 23.

3.5. Dynamic characteristics of the flat convection layer

We will further be concerned with the stress in the layer caused by the flow of the bubbles. This quantity is equal to the specific (per unit surface in unit time) averaged value of the difference between the momenta at the upper and lower boundaries.

Assuming the number of the bubbles in unit volume to be n, and their mean velocity v_D, determine the stress caused by the flow of the bubbles:

$$\tau_b = -4\omega_l l \langle m_\Delta n v_D \rangle = 4\omega_l l A. \qquad (31)$$

The stress that springs up is compensated by the frictional force (Loitsyansky 1987)

$$\tau_f = 2\mu\; def\; v(R) = \mu R \frac{d\omega_g(R)}{dR}. \qquad (32)$$

Here the dynamic friction coefficient μ is the sum of the coefficients of molecular and turbulent friction. The dynamic coefficient of molecular friction depends on temperature which is determined by the distance r to the center of the star. In accordance with the criterion of Richardson (Loitsyansky 1987), initiation of forced convection and turbulence accompanying it in deformation sliding of gas layers seems to be inevitable. The dynamic coefficient of turbulent friction depends on the rate of deformation. It does not seem possible to take account of the ratio of the molecular and turbulent components of friction and their relation to other parameters. Assume the dynamic coefficient of friction to be constant, $\mu = Const.$

Taking into account that $\tau_b = \tau_f$ and taking $\omega_l(r)$ from (4), obtain the differential equation

$$\mu R \frac{d\omega_g(R)}{dR} = 2lAR\frac{d\omega_g(R)}{dR} + 4lA\omega_g(R). \qquad (33)$$

After elementary transformation it assumes the form

$$\frac{d\omega_g(R)}{\omega_g(R)} = \frac{2h}{1-h}\,\frac{dR}{R} = p\frac{dR}{R}, \qquad (34)$$

where

$$h = \frac{2lA}{\mu} = \frac{2lq}{\mu c_p T_e}, \quad p = \frac{2h}{1-h}. \qquad (35)$$

The solution of equation (34) is

$$\omega_g(R) = \left(\frac{R}{r_l}\right)^p; \; \omega_S \approx \left(1 + p\frac{\Delta R}{r_l}\right)\omega_S, \qquad (36)$$

where $\Delta R = R - r_1$, ω_S is the angular velocity of the central non-convective part of the Sun rotating as a solid body, the thickness of the convective layer was considered to be small. Thus, the angular rotational velocity of the surface in the equatorial region of the Sun differs from the angular velocity of its central non-convective part by $\Delta\omega_g \approx p\frac{\Delta r}{r_2}\,\omega_S$.

At a very large friction coefficient ($\mu >> 2lA, h \approx 0,\; p \approx 0$) the velocity is practically independent of the radius ($\omega_g \cong Const$) and the rotation is close to that of a solid body. In the other extreme case where the coefficient of friction is small ($\mu << 2lA$, $h \approx \infty$, $p \approx -2$), the rotation is close to vortex-free, at which the angular velocity of the convective layer is inversely proportional to the square of radius, the linear velocity is inversely proportional to the radius, and the angular velocity of matter in the layer is equal to zero:

$$\omega_g(R) = \left(\frac{R}{r_1}\right)^{-2} \omega_S; \qquad (37)$$
$$v(R) = \left(\frac{R}{r_1}\right)^{-1} v(r_1); \quad \omega_l(R) = 0.$$

Here, when going from the first to the second and third formulae, (2) is used. In the intermediate case, where $\mu > 2lA, p > 0$, the angular velocity of the layer increases with increasing radius, while in the opposite relation ($\mu < 2lA$) it decreases. When $\mu = 2lA$ a sadden change in the coefficient p from infinite positive to infinite negative values takes place. The extremely large jumps in p are not quite correct physically and appeared because of assuming the friction coefficient to be constant and independent of the rate of deformation. In more accurate calculations the dependence of the turbulent friction coefficient on the rate of deformation should be taken into account. In so doing, instead of (34), we come to a non-linear differential equation. Within the framework of this paper we restrict ourselves to consideration of the constant $\mu > 2lA$.

4. Spatial model of convection

We have discussed a plane problem corresponding to the equatorial section of the Sun (latitude $\psi = 0$). In the general case (Fig.1, 5), at some latitude $\psi \neq 0$, we fix a surface outlined by the vector-radius as it rotates from the center of the Sun. Inside this conic surface the bubble rotates together with the layer of the surrounding gas in a latitude circle. The buoyancy force affects the bubble along the corresponding vector-radius. The angular velocity vectors of the bubble, of the surrounding gas, as well as of the Sun, are directed to the pole. Resolve the linear radial velocity v_D of the bubble into components $v_{Ds} = v_D \, Sin\psi$ and $v_{Dc} = v_D \, Cos\psi$ in the directions to the pole and orthogonal to the polar axis. The movement of the bubble in the direction parallel to the polar axis does not give rise to lateral forces. The movement in the orthogonal direction causes the above-discussed effect. Thus, in the case of arbitrary latitude it suffices to substitute the linear velocity v_D of the bubble and the path l in the previous formulae by the expressions of their projections $v_{Dc} = v_D \, Cos\psi$ and $lCos\psi$, respectively. In the end, obtain a common formula

$$\tau = -\,4\omega_l l\, Cos\psi \,\langle n\, m_\Delta v_D\rangle = 4\omega_l lA\, Cos\psi. \qquad (38)$$

Although formula (38) corresponding to a flat model of turbulent friction and low latitudes does not take account of turbulent friction of convective layers at high heliocentric latitudes, for the purpose of simplification we apply it to the entire range of latitudes $\psi \leq 90^0$.

$$\frac{d\omega_g(R)}{\omega_g(R)} = \frac{4lA\, Cos\psi}{\mu - 2lACos\psi}\,\frac{dR}{R} = \qquad (39)$$
$$= \frac{2h\, Cos\psi}{1 - hCos\psi}\,\frac{dR}{R} = p(\psi)\,\frac{dR}{R},$$

where

$$p(\psi) = \frac{2h\, Cos\psi}{1 - hCos\psi}; \quad h = \frac{2lA}{\mu} = \frac{2lq}{\mu c_p T_e}; \qquad (40)$$

c_p, T_e are the averaged parameters of the convective layer.

The solution to equation (39) has the form

$$\omega_g(R) = \left(\frac{R}{r_1}\right)^{p(\psi)}; \qquad (41)$$
$$\omega_S \approx \left(1 + p(\psi)\frac{\Delta R}{r_1}\right)\,\omega_S.$$

According to (41), the angular velocity of the surface of the Sun is $\omega_g(r_2) \equiv \varepsilon\ [^\circ/24^h]$:

$$\varepsilon(\psi) \approx \left(1 + p(\psi)\frac{\Delta r}{r_1}\right)\,\omega_S. \qquad (42)$$

Formula (42) has a clear sense. The surface of the Sun rotates faster than its solid-body nucleus. The difference in velocities between the surface and the nucleus increases with increasing density of the heat flow q, thickness of the convective layer and also path l covered by the bubbles from the moment of their origin to dissipation, and also with decreasing turbulent friction coefficient μ. The angular velocity of the Sun's surface $\varepsilon(\psi)$ decreases with increasing heliocentric latitude ψ in full agreement with observational data.

The results of the calculations by (42) are presented in Table 1 and Fig. 6. The thickness of the convective layer was considered to be equal to a quarter of the radius of the Sun ($\frac{\Delta r}{r_1} = \frac{1}{3}$), the angular velocity of the nucleus of the Sun rotating as a solid body, $\omega_S = 9.83\,^\circ/24^h$, and the parameter h=0.37 were chosen so that a fit to (2) is ensured at the ends of the interval (at the poles and equator). The theoretical relationship corresponds to experimental. The difference is no more than 8% between the rotation velocities at the equator and poles ($3.87^\circ/24^h$) and 2% in the rotation velocity of the surface of the Sun in the region of the equator ($13.7^\circ/24^h$)

Table 1: *The angular velocity of the Sun's surface rotation* ($°/24^h$) *as a function of heliocentric latitude* ψ

ψ(deg)	0	10	20	30	40	50	60	70	80	90
Exper. (1)	13,7	13,7	13,5	13,2	12,7	12,0	11,2	10,5	10,0	9,83
Theory (40)	13,7	13,5	13,3	12,9	12,4	11,8	11,3	10,8	10,2	9,83

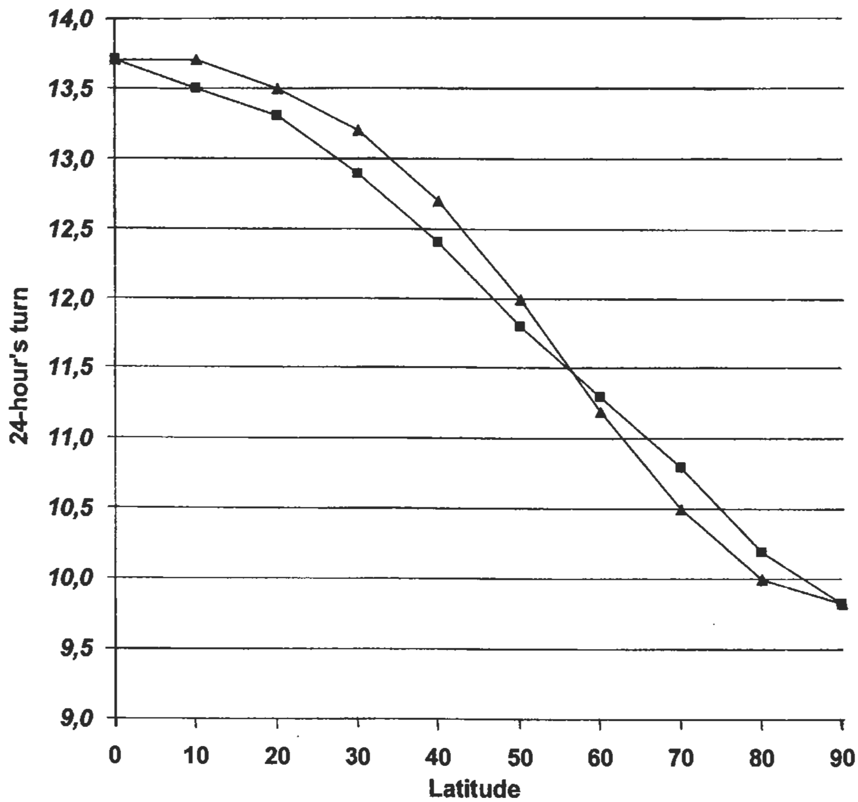

Figure 6: *Differential rotation of the surface of the Sun.*

5. Conclusions

It is assumed that convection in the upper layers of the Sun is realized mainly by the gaseous bubbles of spherical shape. Originally, a two-dimensional model of the phenomenon was developed. The movement of gaseous bubble is interpreted as a shift of the local process in which the bubble plays the part of a solid nucleus of spherical shape. Being in a layer of whirling gas, the nucleus rotates about its axis and is concurrently involved in the radial movement. The nucleus is affected by the Zhukovsky (Magnus) and Coriolis forces. The combined action of these forces causes the speed of the bubble to rise in the direction of rotation of the Sun as a solid. In the course of dissipation of the bubbles, acceleration of rotation of the outer regions with respect to the central part occurs.

In the three-dimensional representation the acceleration in the equatorial region of the surface of the Sun is higher than in the polar regions. In accordance with the relations derived the differential rotation effect rises with increasing density of the heat flow in convective heat transfer, distance covered by the bubbles from the moment of their origin to dissipation, and depth of the convective layer on the Sun. With increasing dynamic friction coefficient, the effect diminishes. The assumptions of thermodynamic spherical isotropy of the Sun, absence of the transitional layer and a sharp boundary between the solid-body rotating central part and the convective layer, absence of friction and involvement of surrounding gas into the bubbles as the latter float up, and also the application of the two-dimensional model at high heliocentric latitudes essentially simplified the discussion. They are not of fundamental character and do not alter the qualitative pattern of the phenomenon, but with their refinement some correction of the relationships and numerical coefficients is possible.

The theoretical results of the present paper are consistent qualitatively and (with appropriate choice of parameters) also quantitatively with the observed pattern of differential rotation of the visible surface of the Sun. Subsequently, it is desirable to substitute actual physical parameters into the derived formulae. Many of them can be estimated on the basis of observations of the globular pattern of the surface of the Sun, refined models of its structure and temperature stratification. The most severe difficulties will, probably, arise with the accuracy of calculation of the friction coefficient. These may be due to both the intricacy of the phenomenon (concurrent existence of forced and free convection) and the known difficulties in the theory of turbulence.

References

Kaplan L.G., 1993, "Local Processes in solid liquid medium and atmosphere". ASOK–press, Stavropol

Kaplan L.G., Nesis E.I., 1998, IFZh, **71**, No. 3, 460

Kaplan L.G., 2000, IFZh, **73**, No. 2, 358

Loitsyansky L.G., 1987, "Mechanics of liquid and gas", M., Nauka

Tossul Zh-L., 1982, "Theory of rotating stars", M., Mir (translation from English)

Vandakurov Yu. V., 1999, ZhTF, **69**, No9, 237